A remark on capillary surfaces in a 3-dimensional space of constant curvature

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Abstract

We generalize a theorem by J. Choe on capillary surfaces for arbitrary 3-dimensional spaces of constant curvature. The main tools in this paper are an extension of a theorem of H. Hopf due to S.-S. Chern and two index lemmas by J. Choe.

1 History

A well known theorem due H. Hopf [4] state that a CMC immersion of the sphere into the Euclidean three-dimensional space is a round sphere. In 1982, in Rio de Janeiro, in occasion of a International Congress at IMPA, S. S. Chern [2] showed a generalization of this theorem when the ambient space has constant sectional curvature. Recently, J. Choe [3], using sufficient hypothesis, generalized Hopf's Theorem for immersion of the closed disk in R^3. Following these lines of ideias, the first result in this paper can be stated as:

Theorem A. *Let S be a CMC immersed compact $C^{2+\alpha}$ surface of disk type in a 3-dimensional ambient space M^3 of constant curvature ($C^{2+\alpha}$ surface means $C^{2+\alpha}$ up to and including ∂S and ∂S is $C^{2+\alpha}$ up to and including its vertices). Suppose that the regular components of ∂S are lines of curvature. If the number of vertices with angle $< \pi$ is less than or equal to 3, then S is totally umbilic.*

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This kind of theorem is motivated by the study of capillary surfaces. In fact, J. Nitsche, in 1995, showed that a regular capillary immersion (see definition 3.1) of the closed disk in the sphere is either a plane disk or a piece of a round sphere. In [3] this result was obtained for capillary immersion without strong regularity assumptions. In 1997 Ros-Souam [7] showed a version of Nitsche's theorem for ambient space with constant sectional curvature. They used Chern's extension of Hopf's theorem. This motivated us to formulate the theorem below:

Theorem B. *Let $U \subset M^3$ be a domain of a 3-dimensional space of constant curvature bounded by totally umbilic surfaces. If S is a capillary surface in U of disk type which is $C^{2+\alpha}$ and S has less than 4 vertices with angle $< \pi$, then S is totally umbilic.*

The paper is organized as follows. In section 2 we recall the context of Chern's work [2], state the main theorems needed here and briefly define the concept of rotation index of the lines of curvature at umbilic points. Finally, we prove the theorems A and B in the section 3.

2 Some Lemmas

In this section we fix some notation and briefly sketch the proof of the main tools used here: S.-S. Chern's generalization of Hopf's theorem and the two index lemmas by J. Choe.

Let M^3 be a 3-dimensional manifold of constant curvature c. Following Chern [2], if $X : S \to M$ is an immersed surface and $p \in S$, we can fix an orthornormal local frame e_1, e_2, e_3 such that e_3 is the unit normal vector to S at x, supposing S orientable ($x \in S$ is a point near to p). If θ_i denotes the coframe ($i = 1, 2, 3$), then $\theta_3 = 0$. The first and second fundamental forms are $I = \theta_1^2 + \theta_2^2$ and $II = h_{11}\theta_1^2 + 2h_{12}\theta_1\theta_2 + h_{22}\theta_2^2$, respectively.

Recall that the invariants $H = \frac{1}{2}(h_{11} + h_{22})$ and $\widetilde{K} = h_{11}h_{22} - h_{12}^2$ are the mean curvature and the total curvature of S, where S has the induced Riemannian metric. By the structure equations (see [2]), we have that the Gaussian curvature is $K = \widetilde{K} + c$. With this setting, we recall the definitions:

Definition 2.1. S is *totally umbilical* (resp. *totally geodesic*) if $II - H \cdot I = 0$ (resp. if $II = 0$).

Defining $\phi = \theta_1 + i\theta_2$, we have a complex structure on S. Note that $II - H \cdot I = \frac{1}{2}(h_{11} - h_{22})(\theta_1^2 - \theta_2^2) + 2h_{12}\theta_1\theta_2$ is a trace zero form and it is the real part of the complex 2-form $\Phi = \widetilde{H}\phi^2$, where $\widetilde{H} = \frac{1}{2}(h_{11} - h_{22}) - h_{12}i$.

Since Φ is uniquely determined by $II - H \cdot I$ and $II - H \cdot I$ is associated to S, Φ is a globally defined 2-form, independent of a choice of local frames.

In the work [4], H. Hopf shows that, in the case $M = \mathbb{R}^3$, i.e., $c = 0$, if the mean curvature is constant, then Φ is holomorphic on S. However, a more general fact is true, as proved by Chern:

Lemma 2.2 (Theorem 1 of Chern [2]). *If $H \equiv$ const., then Φ is a holomorphic 2-form on S.*

The holomorphicity of Φ was used by Hopf to prove that an immersed sphere $f : S^2 \to \mathbb{R}^3$ of *constant mean curvature* (CMC) is round. Indeed, this follows from a standard result about Riemman surfaces which says that, except by the trivial 2-form $\Phi = 0$, there is no holomorphic 2-form on a compact Riemman surface of zero genus. With the same argument, as a corollary of 2.2, Chern was able to conclude that:

Corollary 2.3 (Theorem 2 of Chern [2]). *If $f : S^2 \to M^3$ is an CMC immersed sphere and M^3 has constant mean curvature then f is totally umbilic.*

On the other hand, in the case of surfaces with boundary (with ambient space $M = \mathbb{R}^3$), Choe extends Hopf's arguments to study *capillary surfaces*. In order to make the ideias of Hopf works in his case, Choe introduce a natural concept of *rotation index* of the lines of curvature at umbilic points (*including boundary points*). Now, as a preliminary work, we consider Choe's notion of rotation index in the context of a general ambient space M^3 of constant curvature.

Consider a point $p \in \partial S$. Let $\psi : D_h \to S$ be a conformal parametrization of a neighborhood of p in S, where $D_h = \{(x, y) \in D : y \geq 0\}$ is a half disk and the diameter l of D_h is mapped into ∂S. Let F be the line field on D_h obtained by pulling back (by ψ) the lines of curvature of S. If $\psi(l)$ is a line of curvature of S, we can extend F to a line field on D by reflection through the diameter l. For simplicity, the extension of F to D is denoted by F. At this point, it is natural define the rotation index of the lines of curvature at an umbilic point $p \in \partial S$ to be *half* of the index of F at $\psi^{-1}(p)$. Clearly this definition is independent of the choice of the parametrization ψ. However, the definition only makes sense if we show that the umbilic points on ∂S are isolated. But this fact follows from an easy argument:

Following Choe [3], the equation of the lines of curvature, in complex coordinates is given by

$$\Im(\Phi) = 0,$$

where $\Im z$ denotes the imaginary part of z.

So the rotation index of the lines of curvature is

$$r = \frac{1}{2\pi}\delta(\arg\phi) = -\frac{1}{4\pi}\delta(\arg\Phi),$$

where δ is the variation as one winds once around an isolated umbilic point p. In particular, if p is an interior point of S (i.e., $p \notin S$) and is a zero of order n of Φ, then $\delta(\arg\Phi) = 2\pi n$. Consequently,

$$r = -\frac{n}{2} \leq -\frac{1}{2}. \tag{1}$$

Suppose now that Φ has a zero (resp., pole) of order $n > 0$ (resp., $-n > 0$) at a boundary umbilic point p. Then,

$$r = \frac{1}{2}\big[-\frac{1}{4\pi}\delta(\arg\Phi)\big] = -\frac{n}{4} \tag{2}$$

With these equations in mind, Choe proves the following lemma, which compares interior umbilic points and boundary umbilic points.

Lemma 2.4. *Let S be a CMC immersed $C^{2+\alpha}$ surface (up to and including the boundary ∂S). Suppose that ∂S consist of $C^{2+\alpha}$ curves (up to and including some possible singular points called vertices). If the regular components of ∂S are lines of curvature, then:*

1. *The boundary umbilic points of S are isolated;*
2. *The boundary umbilic points which are not vertices have, at most, rotation index $-1/4$;*
3. *The vertices of S with angle $< \pi$ have rotation index $\leq 1/4$ and the vertices with angle $> \pi$ have rotation index $\leq -1/4$.*

The proof of this lemma is a straightforward consequence (with only minor modifications) of Choe's proof of lemma 2 in [3].

Now we are in position to prove the man results of this paper.

3 Proof of the theorems

Proof of theorem A. Fix $\psi : D \to S$ a conformal parametrization and F the pull-back under ψ of the lines of curvature of S. Since ∂S are lines of curvature, we can apply the Poincaré-Hopf theorem (even in the case that

ψ is a parametrization of a boundary point) to conclude that, if the number of singularities of F is finite, the sum of rotation indices is equal to 1. Let A be the set of such singularities. Suppose that A is finite. Using equation 1, lemma 2.4 and, by hypothesis, the number of vertices with angle is $< \pi$ is ≤ 3, we get the estimate:

$$\sum r(p) \leq 3/4,$$

a contradiction with Poincaré-Hopf's theorem.

Therefore, A is infinite. In particular, since the number of vertices is finite, we have an infinite set $A - \{\text{vertices}\} \subset \{\text{zeros of } \Phi\}$. But Φ is holomorphic. In particular, this implies that $A = S$, so S is totally umbilic. □

We point out that, as remarked by Choe [3], Remark 1, the condition on the number of vertices with angle $< \pi$ is necessary. In fact, a rectangular region in a cylinder $N = \mathbb{S} \times \mathbb{R}^1 \subset \mathbb{R}^3$ bounded by two straight lines and two circles provides a counter-example with 4 vertices with angle $\pi/2$ and rotation index $1/4$.

Before starting the proof of the second main result, we recall the definition:

Definition 3.1. A *capillary surface* S in a domain U of a 3-dimensional space M^3 of constant curvature is a CMC immersed surface which meets ∂U along ∂S at a constant angle.

As a immediate consequence of theorem A, we have the theorem B:

Proof of theorem B. This follows from theorem A and the Terquem - Joachimsthal theorem [8] that says:

"If $C = S_1 \cap S_2$ is a line of curvature of S_1, then C is also a line of curvature of S_2 if and only if S_1 intersect S_2 at a constant angle along C." □

A result for capillary *hypersurfaces* with the same flavor of theorem B was also obtained by Choe (see [3], theorem 3). However, these arguments does not work *a priori* for more general ambient spaces than $\mathbb{R}^n$ since the following fact (valid only in $\mathbb{R}^n$) is used: if X denotes the position vector on S from a fixed point and $\mathbf{H}$ is the mean curvature vector, then $\Delta X = \mathbf{H}$. In particular, it is an open question if there exists *unbalanced* capillary hypersurfaces in the conditions of theorem 3 of Choe.[1]

[1] J. Choe pointed out to one of the authors that, in fact, there exists a generalization of theorem 3 of [3] to be published elsewhere.

To finish this paper we point out that another kind of generalization of the theorem 3 of Choe cited above is obtained by replacing the mean curvature by the *higher order curvatures* H_r. In this direction, Choe showed in [3], theorem 4, that if an immersed hypersurface $S \subset \mathbb{R}^{n+1}$ has constant mean curvature and H_r is constant for some $r \geq 2$, then S is a hypersphere. Moreover, we can replace the constancy of the mean curvature by S is embedded, as Ros proved [6]. Furthermore, for general ambients of constant curvature and supposing only that H_r/H_l is constant, Koh-Lee [5] were able to get the same result. Recently, Alencar-Rosenberg-Santos [1] proved a result in this direction with an extra hypothesis on the Gauss image of S (with ambient space $\mathbb{S}^{n+1}$).

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